Problem 1

In the Black-Scholes-Merton Model the price, c, at time $t = 0$ of a European call option with strike K on a stock with price S_0 at time $t = 0$ and expiry at $T > 0$ is given by:

$$c = S_0 N(d_1) - K\mathsf{e}^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln S_0/K + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

1. What are the assumptions behind this model?

2. Using the call-put-parity derive a formula for the pricing of a put option on the same stock and with same strike and expiry as the call.

3. How can the call price formula be modified to cover the case of a dividend paying stock?

Answer 1

1. The discussion should at least cover

   - the process that the underlying stock is assumed to follow, including the constant parameters of the Brownian Geometric Motion, and that no payments (dividends etc) are implicitly assumed
   - a constant risk free interest rate,
   - that the call price is derived using an arbitrage equilibrium argument

2. Given no dividends note that a portfolio of a long position in one call and a short in one put assuming no arbitrage is equivalent to a forward contract on the stock with forward price equal to the strike. With these assumption the forward contract has a current value equal to the price of the stock less the discounted value of the strike. Thus the put price is

$$\begin{aligned} p =& S_0 N(d_1) - K\mathsf{e}^{-rT} N(d_2) - S_0 + K\mathsf{e}^{-rT} \\ =& - S_0(1 - N(d_1)) + K\mathsf{e}^{-rT}(1 - N(d_2)) \\ =& K\mathsf{e}^{-rT} N(-d_2) - S_0 N(-d_1) \end{aligned}$$

3. In Hull there are described two ways to include dividends in the analysis.

   - You may consider the current stock price as consisting of two parts: The present value of dividends known to be paid until the expiry of the option and the remainder. The last part can be treated as the stock price in the formula for the European call. In principle this reduced stock price should have a volatility that is adjusted upwards if calculated from historical return data (but not if from consistent implicit volatilities of options with same expiry), Hull section 14.12.

- You may also consider dividends as a continuously paid yield q (a special case of the above), Hull section 16.3, and substitute for S_0 in the above the reduced

$$S_0 \mathsf{e}^{-qT}$$

Problem 2

1. For a general derivative with the value $V(S, t)$ contingent on the price of a stock S, which can be assumed to follow an Ito process, and time t define the following Greeks: Delta, Theta and Gamma.

2. Explain the term "Delta-neutral". What could be the purpose of a Delta-neutral portfolio?

3. Suppose a portfolio of the stock and/or derivatives of that stock is Delta-neutral, and that there are no arbitrage possibilities. What can we say about the relation between the Theta and Gamma of the portfolio?

Answer 2

1. The general definition of Delta, Gamma and Theta is given Hull section 18.4.

2. Delta-hedging is discussed in the same section. The discussion should note that

   - a delta neutral portfolio has a value that is unchanged for instantaneous, infinitesimal changes in the price of the stock
   - that this could be used to hedge a portfolio, e.g. of options,
   - but that this would need to be a dynamic hedge

3. For a portfolio with value $\Pi(S, t)$ dependent on a non-dividend-paying stock we have by no arbitrage the Black-Scholes-Merton PDE:

   $$\frac{\partial \Pi}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} + rS\frac{\partial \Pi}{\partial S} - r\Pi = 0$$

   or

   $$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma + rS\Delta = r\Pi$$

   When delta is zero you have

   $$\Theta + \tfrac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

   One interpretation from this is that a delta-neutral, positive-gamma portfolio barring arbitrage will have (for small r) a negative theta (time-decay), cf. Hull, section 18.7.

Problem 3

1. Suppose that the probability that a company is not in default on it's obligations at or before time t is given by $V(t)$. How would we define the hazard rate (or default intensity)?

2. How, and under which assumptions, may we estimate the hazard rate from the interest rate spread on bonds issued by the company? Under what probability measure would we say this estimate is derived? Compare this to a hazard rate that is derived from default frequencies and recovery ratios published by a ratings agency.

3. In the Merton-model the value of a claim on a company is modeled taking credit risk into account by using a variation on the Black-Scholes-Merton option model. Describe the characteristics of the Merton-model and discuss it's potential shortcomings.

Answer 3

1. Given $V(t)$ the hazard rate is the rate of decay of survivors. Assuming differentiability you may define it as

   $$\lambda(t) = -\frac{\frac{\partial V(t)}{\partial t}}{V(t)}$$

   This can be motivated by considering a small discrete time step and letting this approach zero.

2. Under a risk adjusted probability measure the hazard rate may be derived from prices of traded assets, e.g. bonds from the company in question. Assuming that the recovery rate R is known you may estimate the average hazard rate as the hazard rate that makes the discounted expected payoff (i.e. taking defaults into account) to be the price of a risk free bond. Assuming the risky bond pays a spread of s over the risk free bond for a given maturity you can put

   $$\bar{\lambda} = \frac{s}{1-R}$$

   You may similarly booth strap a hazard rate structure from a term structure of credit spreads.

   This analysis should, however, be seen as conducted under a risk neutral ("Q") measure, so that (λ, R) will not be the same as frequencies and averages published by rating agencies (Hull section 23.4-5).

3. In the Merton-model the value of risky debt issued by a limited liability entity is modeled as the value of the assets of the entity less the value of the equity held by the owners of the company. Due to the limited liability the owners can be seen to hold a call option on the assets.

Assuming there is only one form of debt, a zero coupon debt of face value D maturing at time T, the Black-Scholes-Merton Model can be applied. The underlying asset for that model would be the value of the entity assets, which is typically unknown, as is it's volatility. From Ito's lemma you can derive a relationship between the equity volatility and the asset volatility. Assuming that the value of the equity and the equity volatility is known form markets, you have two equations in two unknowns that can be solved numerically for the value of the debt (Hull, section 23.6).

A problem with Merton's model is that values are assumed to follow Ito-processes, i.e. with continuous sample paths. This makes the likelihood of a default a short time step ahead very small (no "jump-to-default"') and credit spreads to go to zero as the maturity of risky bonds goes to zero, which is not in line with empirical evidence (which may be due e.g. to asset values that fundamentally jump in case of default or to asymmetric information between owners and creditors on the value of assets, revealed at default).